

ON DIVISIBILITY OF SUMS OF APÉRY POLYNOMIALS

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ABSTRACT. For any positive integers m and α , we prove that

$$\sum_{k=0}^{n-1} \epsilon^k (2k+1) A_k^{(\alpha)}(x)^m \equiv 0 \pmod{n},$$

where $\epsilon \in \{1, -1\}$ and

$$A_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n}{k}^{\alpha} \binom{n+k}{k}^{\alpha} x^k.$$

1. INTRODUCTION

The Apéry number A_n is defined by

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2.$$

Those numbers play an important role in Apéry's ingredient proof [4] of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$. In 2000, Ahlgren and Ono [1] solved a conjecture of Beukers [2] and showed that for odd prime p ,

$$A_{(p-1)/2} \equiv a(p) \pmod{p^2},$$

where $a(n)$ is the Fourier coefficient of q^n in the modular form $\eta(2z)^4 \eta(4z)^4$.

Recently, Sun [7] defined the Apéry polynomial as

$$A_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 x^k,$$

and proved several new congruences for the sums of $A_n(x)$. For examples,

$$\sum_{k=0}^{n-1} (2k+1) A_k(x) \equiv 0 \pmod{n}$$

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for every positive integer n . In fact, he showed that

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k+1) A_k(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k} \binom{n+k}{2k+1} \binom{2k}{k} x^k.$$

Furthermore, Sun proposed the following conjecture.

Conjecture 1.1. For $m \in \{1, 2, 3, \dots\}$,

$$\sum_{k=0}^{n-1} \epsilon^k (2k+1) A_k(x)^m \equiv 0 \pmod{n}, \quad (1.1)$$

where $\epsilon \in \{1, -1\}$.

In [3], Guo and Zeng proved that

$$\frac{1}{n} \sum_{k=0}^{n-1} (-1)^k (2k+1) A_k(x) = (-1)^{n-1} \sum_{k=0}^{n-1} \binom{2k}{k} x^k \sum_{j=0}^k \binom{k}{j} \binom{k+j}{j} \binom{n-1}{k+j} \binom{n+k+j}{k+j}.$$

On the other hand, in [6], Sun also define the central Delannoy polynomial

$$D_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} x^k.$$

He showed that

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k+1) D_k(x) = \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n+k}{k} x^k.$$

Sun also conjectured that

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k+1) D_k(x)^m$$

is always an integer.

In fact, motivated by [5] and [1, eq. (1.7)], we may define the generalized Apéry polynomial

$$A_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n}{k}^\alpha \binom{n+k}{k}^\alpha x^k,$$

where α is a positive integer. (In [3], Guo and Zeng called such polynomial as the Schmidt polynomial.) In the same paper, Guo and Zeng also proved that fact all $\alpha \geq 1$, there exist explicit formulas for

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k+1) A_k^{(\alpha)}(x) \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k (2k+1) A_k^{(\alpha)}(x).$$

However, no explicit formula is known for

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k+1) A_k^{(\alpha)}(x)^m \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k (2k+1) A_k^{(\alpha)}(x)^m$$

when $m \geq 2$.

In this paper, we shall prove

Theorem 1.1. *For any positive integers m and α ,*

$$\sum_{k=0}^{n-1} (2k+1) A_k^{(\alpha)}(x)^m \equiv 0 \pmod{n}, \quad (1.2)$$

and

$$\sum_{k=0}^{n-1} (-1)^k (2k+1) A_k^{(\alpha)}(x)^m \equiv 0 \pmod{n}. \quad (1.3)$$

In the next sections, we shall use q -congruences to prove (1.2) and (1.3) respectively.

2. PROOF OF (1.2)

For an integer n , define the q -integer

$$[n]_q = \frac{1 - q^n}{1 - q}.$$

Clearly $\lim_{q \rightarrow 1} [n]_q = n$. For a non-negative integer k , the q -binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{1 \leq j \leq k} [n - j + 1]_q}{\prod_{1 \leq j \leq k} [j]_q}.$$

In particular, $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$. Also, we set $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k < 0$. It is easy to see that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q with integral coefficients, since

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

Below we introduce the notion of q -congruences. Suppose that a, b, n are integers and $a \equiv b \pmod{n}$. Then over the polynomial ring $\mathbb{Q}(q)$, we have

$$\frac{1 - q^a}{1 - q} - \frac{1 - q^b}{1 - q} = q^a \cdot \frac{1 - q^{b-a}}{1 - q} \equiv 0 \pmod{\frac{1 - q^n}{1 - q}},$$

i.e., $[a]_q \equiv [b]_q \pmod{[n]_q}$. Furthermore, for the q -binomial coefficients, we have the following q -Lucas congruence.

Lemma 2.1. *Suppose that $d > 1$ is a positive integer. Suppose that a, b, h, l are integers with $0 \leq b, l \leq d - 1$. Then*

$$\begin{bmatrix} ad + b \\ hd + l \end{bmatrix}_q \equiv \begin{pmatrix} a \\ h \end{pmatrix} \begin{bmatrix} b \\ l \end{bmatrix}_q \pmod{\Phi_d(q)},$$

where $\Phi_d(q)$ is the d -th cyclotomic polynomial.

Define the generalized q -Apéry polynomial

$$A_k^{(\alpha)}(x; q) = \sum_{j=0}^k q^{\binom{j}{2} - jk} \begin{bmatrix} k \\ j \end{bmatrix}_q^\alpha \begin{bmatrix} k + j \\ j \end{bmatrix}_q^\alpha x^j.$$

In order to prove (1.2), it suffices to show that

Theorem 2.1.

$$\sum_{k=0}^{n-1} q^{n-1-k} [2k+1]_q A_k^{(\alpha)}(x; q) \equiv 0 \pmod{[n]_q}. \quad (2.1)$$

Let us explain why (2.1) implies (1.2)). Since $[n]_q$ is a primitive polynomial (i.e., the greatest divisor of all coefficients of $[n]_q$ is 1), by the Gauss lemma, there exists a polynomial $H(x, q)$ with integral coefficients such that

$$\sum_{k=0}^{n-1} q^{n-1-k} [2k+1]_q A_k^{(\alpha)}(x; q) = [n]_q H(x, q). \quad (2.2)$$

Substituting $q = 1$ in (2.2), we get (1.2).

It is not difficult to check that

$$[n]_q = \prod_{\substack{d|n \\ d>1}} \Phi_d(q).$$

The advantage of q -congruences is that we only need to prove that

$$\sum_{k=0}^{n-1} q^{n-1-k} [2k+1]_q A_k^{(\alpha)}(x; q) \equiv 0 \pmod{\Phi_d(q)}$$

for every divisor $d > 1$ of n . Note that

$$\begin{aligned} \begin{bmatrix} k+j \\ j \end{bmatrix}_q &= \frac{(1-q^{k+1})(1-q^{k+2}) \cdots (1-q^{k+j})}{(1-q)(1-q^2) \cdots (1-q^j)} \\ &= (-1)^j \frac{q^{jk + \binom{j+1}{2}} (1-q^{-k-1})(1-q^{-k-2}) \cdots (1-q^{-k-j})}{(1-q)(1-q^2) \cdots (1-q^j)} \\ &= (-1)^j q^{jk + \binom{j+1}{2}} \begin{bmatrix} -k-1 \\ j \end{bmatrix}_q. \end{aligned}$$

So

$$A_k^{(\alpha)}(x; q) = \sum_{-\infty < j < +\infty} (-1)^{\alpha j} q^{\alpha j^2} \begin{bmatrix} k \\ j \end{bmatrix}_q^\alpha \begin{bmatrix} -k-1 \\ j \end{bmatrix}_q^\alpha x^j.$$

Suppose that $d > 1$ is a divisor of n . Let $h = n/d$. Write $k = ad + b$ where $0 \leq b \leq d-1$. Then by Lemma 2.1,

$$\begin{aligned} & A_{ad+b}^{(\alpha)}(x; q) \\ &= \sum_{\substack{-\infty < s < +\infty \\ 0 \leq t \leq d-1}} (-1)^{\alpha(sd+t)} q^{\alpha(sd+t)^2} \begin{bmatrix} ad+b \\ sd+t \end{bmatrix}_q^\alpha \begin{bmatrix} (-a-1)d + d - b - 1 \\ sd+t \end{bmatrix}_q^\alpha x^{sd+t} \\ &\equiv \sum_{\substack{-\infty < s < +\infty \\ 0 \leq t \leq d-1}} (-1)^{\alpha(sd+t)} q^{\alpha t^2} \binom{a}{s} \begin{bmatrix} b \\ t \end{bmatrix}_q^\alpha \binom{-a-1}{s} \begin{bmatrix} d-b-1 \\ t \end{bmatrix}_q^\alpha x^{sd+t} \pmod{\Phi_d(q)}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{k=0}^{n-1} [2k+1]_q q^{n-1-k} A_k^{(\alpha)}(x; q)^m &= \sum_{\substack{0 \leq a \leq h-1 \\ 0 \leq b \leq d-1}} q^{hd-1-ad-b} [2ad+2b+1]_q A_{ad+b}^{(\alpha)}(x; q)^m \\ &\equiv \sum_{\substack{0 \leq a \leq h-1 \\ 0 \leq b \leq d-1}} q^{-1-b} [2b+1]_q B_{a,b,d}^{(\alpha)}(x; q)^m \pmod{\Phi_d(q)}, \end{aligned}$$

where

$$B_{a,b,d}^{(\alpha)}(x; q) = \sum_{\substack{-\infty < s < +\infty \\ 0 \leq t \leq d-1}} (-1)^{\alpha(sd+t)} q^{\alpha t^2} \binom{a}{s} \begin{bmatrix} b \\ t \end{bmatrix}_q^\alpha \binom{-a-1}{s} \begin{bmatrix} d-b-1 \\ t \end{bmatrix}_q^\alpha x^{sd+t}.$$

Similarly, since $k = ad + b \iff n - k - 1 = (h - a - 1)d + (d - b - 1)$ and $B_{a,b,d}^{(\alpha)}(x; q) = B_{a,d-b-1,d}^{(\alpha)}(x; q)$, we have

$$\begin{aligned} \sum_{k=0}^{n-1} q^k [2n-2k-1]_q A_{n-k-1}^{(\alpha)}(x; q)^m &\equiv \sum_{\substack{0 \leq a \leq h-1 \\ 0 \leq b \leq d-1}} q^b [-2b-1]_q B_{h-a-1,d-b-1,d}^{(\alpha)}(x; q)^m \\ &\stackrel{a'=h-a-1}{=} \sum_{\substack{0 \leq a' \leq h-1 \\ 0 \leq b \leq d-1}} q^b [-2b-1]_q B_{a',b,d}^{(\alpha)}(x; q)^m \pmod{\Phi_d(q)}. \end{aligned}$$

Note that

$$q^{-1-b} [2b+1]_q + q^b [-2b-1]_q = q^{-1-b} - q^b + q^b - q^{-b-1} = 0.$$

Therefore,

$$\begin{aligned}
& 2 \sum_{k=0}^{n-1} q^{n-1-k} [2k+1]_q A_k^{(\alpha)}(x; q)^m \\
&= \sum_{k=0}^{n-1} q^{n-1-k} [2k+1]_q A_k^{(\alpha)}(x; q)^m + \sum_{k=0}^{n-1} q^k [2n-2k-1]_q A_{n-1-k}^{(\alpha)}(x; q)^m \\
&\equiv 0 \pmod{\Phi_d(q)}.
\end{aligned}$$

This concludes the proof of Theorem 2.1.

3. PROOF OF (1.3)

The proof of (1.3) is a little complicated.

Theorem 3.1.

$$\sum_{k=0}^{n-1} (-1)^k q^{n-1-k} [2k+1]_q A_k^{(\alpha)}(x; q^2)$$

is divisible by

$$\prod_{\substack{d|n \\ d>1 \text{ is odd}}} \Phi_d(q) \cdot \prod_{\substack{d|n \\ d \text{ is even}}} \Phi_d(q^2).$$

Clearly we only need to prove that

$$\sum_{k=0}^{n-1} (-1)^k q^{n-1-k} [2k+1]_q A_k^{(\alpha)}(x; q^2)^m + \sum_{k=0}^{n-1} (-1)^{n-1-k} q^k [2n-2k-1]_q A_{n-1-k}^{(\alpha)}(x; q^2)^m.$$

is divisible by $\Phi_d(q)$ for odd $d > 1$ and by $\Phi_d(q^2)$ for even d respectively.

Lemma 3.1. *If $d > 1$ is odd, then $\Phi_d(q)$ divides $\Phi_d(q^2)$. If d is even, then $\Phi_d(q^2) = \Phi_{2d}(q)$.*

Proof. We know that for $d > 1$,

$$\Phi_d(q) = \prod_{\xi \text{ is } d\text{-th primitive root of unity}} (q - \xi).$$

Suppose that d is odd and ξ is an arbitrary d -th primitive root of unity. Then ξ^2 also is a d -th primitive root of unity, i.e., $\Phi_d(\xi^2) = 0$. Hence $\Phi_d(q)$ divides $\Phi_d(q^2)$. Similarly, if d is even and ξ is a $2d$ -th primitive root of unity, then ξ^2 is a d -th primitive root of unity. So $\Phi_{2d}(q)$ divides $\Phi_d(q^2)$. Note that now $\deg \Phi_{2d} = \phi(2d) = 2\phi(d) = 2 \deg \Phi_d$, where ϕ is the Euler totient function. We must have $\Phi_d(q^2) = \Phi_{2d}(q)$. \square

Suppose that $d > 1$ is an odd divisor of n . Let $h = n/d$. Then

$$\begin{aligned}
& \sum_{k=0}^{n-1} (-1)^k q^{n-1-k} [2k+1]_q A_k^{(\alpha)}(x; q^2)^m \\
& \equiv \sum_{\substack{0 \leq a \leq h-1 \\ 0 \leq b \leq d-1}} (-1)^{ad+b} q^{hd-1-ad-b} [2(ad+b)+1]_q B_{a,b,d}^{(\alpha)}(x; q^2)^m \pmod{\Phi_d(q^2)} \\
& \equiv \sum_{\substack{0 \leq a \leq h-1 \\ 0 \leq b \leq d-1}} (-1)^{ad+b} q^{-1-b} [2b+1]_q B_{a,b,d}^{(\alpha)}(x; q^2)^m \pmod{\Phi_d(q)}.
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{k=0}^{n-1} (-1)^{n-1-k} [2n-2k-1]_q q^k A_{n-1-k}^{(\alpha)}(x; q^2)^m \\
& \equiv \sum_{\substack{0 \leq a \leq h-1 \\ 0 \leq b \leq d-1}} (-1)^{hd-1-ad-b} q^{ad+b} [2sd-2(ad+b)-1]_q B_{h-a-1,d-b-1,d}^{(\alpha)}(x; q^2)^m \\
& \equiv \sum_{\substack{0 \leq a' \leq h-1 \\ 0 \leq b \leq d-1}} (-1)^{a'd+d-1-b} [-2b-1]_q q^b B_{a',b,d}^{(\alpha)}(x; q^2)^m \pmod{\Phi_d(q)}.
\end{aligned}$$

Since d is odd,

$$(-1)^{ad+b} q^{-1-b} [2b+1]_q + (-1)^{ad+d-1-b} q^b [-2b-1]_q = 0.$$

So $\Phi_d(q)$ divides

$$\sum_{k=0}^{n-1} (-1)^k q^{n-1-k} [2k+1]_q A_k^{(\alpha)}(x; q^2)^m + \sum_{k=0}^{n-1} (-1)^{n-1-k} q^k [2n-2k-1]_q A_{n-1-k}^{(\alpha)}(x; q^2)^m.$$

Suppose that d is an even divisor of n . Then

$$\begin{aligned}
& \sum_{k=0}^{n-1} (-1)^k q^{n-1-k} [2k+1]_q A_k^{(\alpha)}(x; q^2)^m \\
& \equiv \sum_{\substack{0 \leq a \leq h-1 \\ 0 \leq b \leq d-1}} (-1)^{ad+b} q^{hd-1-(ad+b)} [2(ad+b)+1]_q B_{a,b,d}^{(\alpha)}(x; q^2)^m \\
& \equiv \sum_{\substack{0 \leq a \leq h-1 \\ 0 \leq b \leq d-1}} (-1)^{ad+b} q^{hd-ad-1-b} [2b+1]_q B_{a,b,d}^{(\alpha)}(x; q^2)^m \pmod{\Phi_d(q^2)}.
\end{aligned}$$

And

$$\begin{aligned}
& \sum_{k=0}^{n-1} (-1)^{n-1-k} q^k [2n-2k-1]_q A_{n-1-k}^{(\alpha)}(x; q^2)^m \\
& \equiv \sum_{\substack{0 \leq a \leq h-1 \\ 0 \leq b \leq c-1}} (-1)^{hd-1-(ad+b)} q^{ad+b} [2hd-2(ad+b)-1]_q B_{h-a-1, d-b-1, d}^{(\alpha)}(x; q^2)^m \\
& \equiv \sum_{\substack{0 \leq a' \leq h-1 \\ 0 \leq b \leq d-1}} (-1)^{a'd+d-1-b} q^{hd-a'd-d+b} [-2b-1]_q B_{a', b, d}^{(\alpha)}(x; q^2)^m \pmod{\Phi_d(q^2)}.
\end{aligned}$$

Note that $\Phi_d(q^2) = \Phi_{2d}(q)$ divides $1 + q^d = (1 - q^{2d})/(1 - q^d)$, i.e.,

$$q^d \equiv -1 \pmod{\Phi_d(q^2)}.$$

We have

$$\begin{aligned}
& (-1)^{ad+b} q^{hd-ad-1-b} [2b+1]_q + (-1)^{ad+d-1-b} q^{hd-ad-d+b} [-2b-1]_q \\
& \equiv (-1)^{ad+b} q^{hd-ad} (q^{-1-b} [2b+1]_q + q^b [-2b-1]_q) \\
& = 0 \pmod{\Phi_d(q^2)}.
\end{aligned}$$

That is, $\Phi_d(q^2)$ divides

$$\sum_{k=0}^{n-1} (-1)^k q^{n-1-k} [2k+1]_q A_k^{(\alpha)}(x; q^2)^m + \sum_{k=0}^{n-1} (-1)^{n-1-k} q^k [2n-2k-1]_q A_{n-1-k}^{(\alpha)}(x; q^2)^m.$$

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